International Journal of Engineering, Science and Mathematics
Vol. 6 Issue 8, December 2017 (Special Issue)
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

Numerical Solution of Weakly Singular Integral Equations using Legendre Wavelets

S. C. Shiralashetti [*a]
R. A. Mundewadi[b] [**b]
S. S. Naregal [***c]
B. Veeresh[d] [****d]

Abstract

In this paper, Legendre wavelet collocation method is developed for the numerical solution of weakly singular integral equations. Properties of Legendre wavelet and its function approximation are discussed. Proposed method converts the integral equation into system of algebraic equations and solving these equations to obtain the Legendre wavelet coefficients. Convergence and error analysis of the Illustrative examples are presented to show the efficiency of the proposed method.

Keywords:

Weakly singular integral equations, Legendre wavelet, Legendre polynomials.

Author correspondence:

S. S. Naregal[***c]
Department of Mathematics, Karnatak University, Dharwad-580003, Karnataka, India
Email: naregal.sharada@gmail.com; Mob.: +91 9448923152

1. Introduction

Weakly singular Fredholm integral equations (WSFIEs) have many applications in mathematical physics. These equations arise in the heat conduction problem posed by mixed boundary conditions, potential problems, Dirichlet problem and radiative equilibrium. Furthermore, some important applications of WSFIEs in the fields of fracture mechanics, elastic contact problems, theory of porous filtering, combined infrared radiation and molecular conduction. It is difficult to solve these equations analytically. Hence, numerical schemes are required for dealing with these equations in a proper manner [2]. Namely, Euler-Maclaurin summation formula [2], Sinc-collocation method [8], Hybrid collocation method [4].

Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation

[*a,b,c] Department of Mathematics, Karnatak University, Dharwad-580003, Karnataka, India

[*d] R. Y. M. Engg. College, Bellary-583104, Karnataka, India

and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [3, 5]. Since from 1991 the various types of wavelet method have been applied for the numerical solution of different kinds of integral equations, a detailed survey found in [7]. Some of the authors have approached weakly singular Fredholm integral equations using wavelet based methods. Such as Legendre wavelets [1] and Trigonometric Hermite wavelet [6]. In this paper, we developed Legendre wavelet collocation method for the numerical solution of weakly singular Fredholm integral equations.
The article is organized as follows: In section 2, the properties of Legendre wavelets are given. In section 3 convergence and error analysis is discussed. In section 4 devoted the method of solution. In section 5, we report our numerical results and demonstrated the accuracy of the proposed scheme. Conclusion is discussed in section 6.

2. Properties of Legendre wavelets

2.1 Wavelets

In recent years, wavelets have found their way into many different fields of science and engineering. Wavelets constitute a family of single functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter *a* and the translation parameter *b* vary continuously, we have the following family of continuous wavelets.

$$\psi_{a,b}(t)=|a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right),\quad a,b\in R,\quad a\neq 0$$

If we restrict the parameters *a* and *b* to discrete values as $a=a_0^{-k}, b=nb_0a_0^{-k}, a_0>1, b_0>0$ and *n*, and *k* positive integers, we have the following family of discrete wavelets,

$$\psi_{k,n}(t)=|a_0|^{k/2}\psi(a_0^k t-nb_0),$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0=2$ and $b_0=1$ then $\psi_{k,n}(t)$ forms an orthogonal basis.

2.2 Legendre wavelets

Legendre wavelets $L_{nm}(t)=L(k,\hat{n},m,t)$ have four arguments; $\hat{n}=2n-1, n=1,2,3,...,2^{k-1}$, k can assume any positive integer, *m* is the order of the Legendre polynomials and t is the normalized time. They are defined on the interval [0, 1) by:

$$L_{n,m}(t)=\begin{cases}\sqrt{m+\frac{1}{2}}\,2^{k/2}l_m(2^k t-\hat{n}), & \frac{\hat{n}-1}{2^k}\leq t<\frac{\hat{n}+1}{2^k},\\ 0, & otherwise.\end{cases}\tag{1}$$

where $m=0,1,2,...,M-1, n=1,2,3,...,2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a=2^{-k}$ and translation parameter is $b=\hat{n}2^{-k}$.

Here, $l_m(t)$ are the well-known Legendre polynomial of order m, which are orthogonal with respect to the weight function $w(t)=1$ and satisfy the following recursive formula,

$$l_0(t)=1,$$
$$l_1(t)=t,$$
$$l_{m+1}(t)=\frac{2m+1}{m+1}tl_m(t)-\frac{m}{m+1}l_{m-1}(t),\quad m=1,2,3,...$$

2.3 Function Approximation

A function $f(t)\in L^2[0,1]$ may be expanded as:

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}L_{n,m}(t),\tag{2}$$

where

$$c_{n,m}=(f(t),L_{n,m}(t)).\tag{3}$$

In (3), (. , .) denotes the inner product.
If the infinite series in (2) is truncated, then (2) can be rewritten as:

$$f(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}L_{n,m}(t)=C^T\Psi(t), \tag{4}$$

where C and $L(t)$ are $2^{k-1}M\times 1$ matrices given by:

$$C=[c_{10},c_{11},...,c_{1,M-1},c_{20},...,c_{2,M-1},...,c_{2^{k-1},0},...,c_{2^{k-1},M-1}]^T$$
$$=[c_1,c_2,...,c_{2^{k-1}M}]^T, \tag{5}$$

and

$$\Psi(t)=[L_{10}(t),L_{11}(t),...,L_{1,M-1}(t),L_{20}(t),...,L_{2,M-1}(t),...,L_{2^{k-1},0}(t),...,L_{2^{k-1},M-1}(t)]^T$$
$$=[L_1(t),L_2(t),...,L_{2^{k-1}M}(t)]^T. \tag{6}$$

3. Convergence and Error Analysis

Theorem 3.1. The series solution of Legendre wavelet $y(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}L_{n,m}(x)$ is converges to $y(x)$.

Proof: Let $L^2(R)$ be the infinite dimensional Hilbert space and $L_{n,m}$ is defined as Eq. (2) forms an orthonormal basis.

Let $y(x)=\sum_{i=0}^{M-1}c_{n,i}L_{n,i}(x)$ where $c_{n,i}=\langle y(x),L_{n,i}(x)\rangle$ for a fixed n.

Let us denote the sequence of partial sums S_n of $\{c_{n,i}L_{n,i}(x)\}$, Let S_n and S_m be the partial sums with $n\geq m$. We have to prove S_n is a Cauchy sequence in Hilbert space $L^2(R)$.

Choose, $S_n=\sum_{i=0}^{n}c_{n,i}L_{n,i}(x)$, Now $\langle y(x),S_n\rangle=\left\langle y(x),\sum_{i=0}^{n}c_{n,i}L_{n,i}(x)\right\rangle=\sum_{i=m+1}^{n}|c_{n,i}|^2$

We claim that $\|S_n-S_m\|^2=\sum_{i=m+1}^{n}|c_{n,i}|^2,\quad \forall n>m$

Now $\left\|\sum_{i=m+1}^{n}c_{n,i}L_{n,i}(x)\right\|^2=\left\langle\sum_{i=m+1}^{n}c_{n,i}L_{n,i}(x),\sum_{i=m+1}^{n}c_{n,i}L_{n,i}(x)\right\rangle=\sum_{i=m+1}^{n}|c_{n,i}|^2,\quad\forall n>m$

thus, $\left\|\sum_{i=m+1}^{n}c_{n,i}L_{n,i}(x)\right\|^2=\sum_{i=m+1}^{n}|c_{n,i}|^2,\quad\forall n>m$

Since, Bessel's inequality, we have $\sum_{i=m+1}^{n}|c_{n,i}|^2\leq\|y(x)\|^2$ is bounded and convergent.

Hence, $\left\|\sum_{i=m+1}^{n}c_{n,i}L_{n,i}(x)\right\|^2\to 0\ \ as\ \ m,n\to\infty.$

This implies, $\left\|\sum_{i=m+1}^{n}c_{n,i}L_{n,i}(x)\right\|\to 0.$ and

Therefore $\{S_p\}$ is a Cauchy sequence and it converges to s (say).

We assert that $y(x)=s$

Now $\langle s-y(x),L_{n,i}(x)\rangle=\langle s,L_{n,i}(x)\rangle-\langle y(x),L_{n,i}(x)\rangle=\langle s,L_{n,i}(x)\rangle-\left\langle\lim_{n\to\infty}S_n,L_{n,i}(x)\right\rangle=0$,

This implies,

$$\langle s-y(x),L_{n,i}(x)\rangle=0$$

Hence $y(x)=s$ and $\sum_{i=0}^{n} c_{n,i} L_{n,i}(x)$ converges to $y(x)$ as $n\to\infty$ and proved.

Theorem 3.2. Suppose that $y(x)\in C^m[0,1]$ and $C^T\Psi(x)$ is the approximate solution using Legendre wavelet. Then the error bound would be given by,

$$\|E(x)\| \le \left\| \frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} \left|y^m(x)\right| \right\|.$$

Proof: Applying the definition of norm in the inner product space, we have,

$$\|E(x)\|^2 = \int_0^1 \left[y(x)-C^T\Psi(x)\right]^2 dx.$$

Divide interval [0, 1] into 2^{k-1} subintervals $I_n = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right], n=1,2,3,...,2^{k-1}.$

$$\|E(x)\|^2 = \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[y(x)-C^T\Psi(x)\right]^2 dx.$$

$$\|E(x)\|^2 \le \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[y(x)-P_m(x)\right]^2 dx.$$

Where $P_m(x)$ is the interpolating polynomial of degree m which approximates $y(x)$ on I_n.
By using the maximum error estimate for the polynomial on I_n, then

$$\|E(x)\|^2 \le \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in I_n} \left|y^m(x)\right|\right]^2 dx.$$

$$\|E(x)\|^2 \le \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} \left|y^m(x)\right|\right]^2 dx.$$

$$\|E(x)\|^2 = \int_0^1 \left[\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} \left|y^m(x)\right|\right]^2 dx$$

$$\|E(x)\| \le \left\| \frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} \left|y^m(x)\right| \right\|.$$

4. Legendre Wavelet Collocation Method of Solution

Consider the Weakly singular Fredholm integral equation,

$$y(x) = f(x) + \int_0^1 \frac{y(t)}{\sqrt{1-x}} dt, \quad 0\le x\, t \le 1 \tag{7}$$

To solve Eq. (7), the procedure is as follows:
Step 1: We first approximate $y(x)$ as truncated series defined in Eq. (4). That is,

$$y(x) = C^T\Psi(x) \tag{8}$$

where C and $\Psi(x)$ are defined similarly to Eqs. (5) and (6).
Step 2: Then substituting Eq. (8) in Eq. (7), we get,

$$C^T\Psi(x) = f(x) + \int_0^1 \frac{C^T\Psi(t)}{\sqrt{1-x}} dt \tag{9}$$

Step 3: Substituting the collocation point $x_i = \frac{2i-1}{2^k M}, i=1,2,...,2^{k-1}M$ in Eq. (9). Then we obtain,

$$C^T\Psi(x_i) = f(x_i) + C^T \int_0^1 \frac{\Psi(t)}{\sqrt{1-x_i}} dt \tag{10}$$

$$C^T(\Psi(x_i)-G)=f,\ \text{where}\ G=\int_0^1 \frac{\Psi(t)}{\sqrt{1-x_i}}dt$$

Step 4: Now, we get the system of algebraic equations with unknown coefficients.

$$C^T K = f,\ \text{where}\ K=(\Psi(x_i)-G)$$

Solving the above system of equations, we get the Legendre wavelet coefficients '*C*' and then substituting these coefficients in Eq. (8), we get the required approximate solution of Eq. (7).

5. Numerical Examples

In this section, we present Hermite wavelet (HW) method for the numerical solution of Fredholm integral equations of the second kind in comparison with existing method to demonstrate the capability of the present method and error analysis are shown in tables and figures. Error function is presented to verify the accuracy and efficiency of the following numerical results:

$$E_{Max} = Error\ function = \|y_e(t_i)-y_a(t_i)\|_2 = \sqrt{\sum_{i=1}^{n}(y_e(t_i)-y_a(t_i))^2}$$

where, y_e and y_a are the exact and approximate solution respectively.

Example 1. Firstly, consider the weakly singular Fredholm integral equation of the second kind,

$$y(x)=x^2-\frac{16}{15}+\int_0^1 \frac{y(t)}{\sqrt{1-t}}dt,\quad 0\le x\le 1. \tag{11}$$

which has the exact solution $y(x)=x^2$. Using Legendre Wavelet Collocation Method presented in section 4 for the solution of Eq. (11) with k =1 and M =3, we obtain, $f=[-1.0389\ \ -0.8167\ \ -0.3722]$,

$$K=\begin{bmatrix} -1.0000 & -1.0000 & -1.0000 \\ -2.3094 & -1.1547 & 0.0000 \\ -0.5217 & -2.0125 & -0.5217 \end{bmatrix}$$

Next, we get the Legendre wavelet coefficients, $C=[0.3333\ \ \ 0.2887\ \ \ 0.0745]$

substituting these coefficients in Eq. (8), we get the accurate solution of Eq. (11) as similar as exact solution $y(x)=x^2$ and the maximum error is 1.11e-016 and compared to the existing method [2] has the maximum error for n = 256 is 5.02e-06. This shows the efficiency of the proposed method.

Example 2. Next, consider [2],

$$y(x)=\sqrt{x}-\frac{\pi}{2}+\int_0^1 \frac{y(t)}{\sqrt{1-t}}dt,\quad 0\le x\le 1. \tag{12}$$

which has the exact solution $y(x)=\sqrt{x}$. We solved the Eq. (12) by approaching the present method for k = 1 and M = 5, we get the approximate solution as shown in table 1 and the maximum error is 2.98e-04.

Table 1 Numerical result of example 2.

x	Exact solution	Present method (k = 1, M = 5)	Absolute Error
0.1	0.3162	0.3159	2.98e-04
0.2	0.4472	0.4445	2.70e-03
0.3	0.5477	0.5474	2.98e-04
0.4	0.6325	0.6328	3.29e-04
0.5	0.7071	0.7068	2.98e-04
0.6	0.7746	0.7738	7.54e-04
0.7	0.8367	0.8364	2.98e-04
0.8	0.8944	0.8950	5.39e-04
0.9	0.9487	0.9484	2.98e-04

Example 3. Lastly, consider [2],

$$y(x) = \exp(x) - 4.0602 + \int_0^1 \frac{y(t)}{\sqrt{1-t}} dt, \quad 0 \le x \le 1. \tag{13}$$

which has the exact solution $y(x) = \exp(x)$. Applying the proposed method to solve Eq. (13) for $k = 1$ and $M = 8$. We obtain the approximate solution $y(x)$ as shown in table 2 and the maximum error is 4.30e-05 as shown in figure 1.

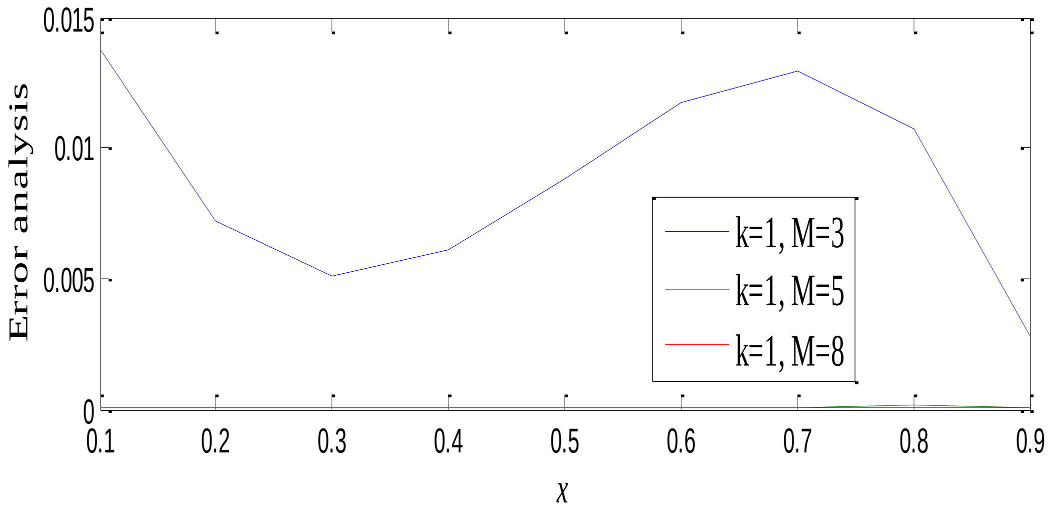

Fig. 1. Error analysis of example 3.

Table 2. Numerical result of example 3.

x	Exact solution	Present method ($k = 1$, $M = 8$)
0.1	1.105170	1.105213
0.2	1.221402	1.221445
0.3	1.349858	1.349901
0.4	1.491824	1.491867
0.5	1.648721	1.648764
0.6	1.822118	1.822161
0.7	2.013752	2.013795
0.8	2.225540	2.225583
0.9	2.459603	2.459646

6. Conclusion

The Legendre wavelet collocation method is applied for the numerical solution of weakly singular Fredholm integral equations of the second kind. The present method reduces an integral equation into a set of algebraic equations. For instance in example 1, our results are higher accuracy with exact ones and existing method [2], subsequently other examples are also same in the nature. The numerical result shows that the accuracy improves with increasing the k and M for better accuracy. Convergence and error analysis justifies the efficiency, validity and applicability of the present technique.

Acknowledgement

Authors acknowledge the support received from the University Grants Commission (UGC), Govt. of India for grant under UGC-SAP DRS-III for 2016-2021:F.510/3/DRS-III/2016(SAP-I) Dated: 29th Feb. 2016.

References

[1] Adibi, H., and P. Assari. 2010. On the numerical solution of weakly singular Fredholm integral equations of the second kind using Legendre wavelets. Journal of Vibration and Control. 17(5): 689–698. DOI: 10.1177/1077546310366865

[2] Behzadi, R., E. Tohidi, and F. Toutounian. 2014. Numerical solution of weakly singular Fredholm integral equations via generalization of the Euler-Maclaurin summation formula. Journal of Taibah university for science. 8: 199-205.

[3] Beylkin, G., R. Coifman and V. Rokhlin. 1991. Fast wavelet transforms and numerical algorithms I. Commun. Pure Appl. Math. 44: 141–183.

[4] Cao, Y., M. Huang, L. Liu, Y. Xu. 2007. Hybrid collocation methods for Fredholm integral equations with weakly singular kernels, Applied Numerical Mathematics 57: 549–561.

[5] Chui, C. K. 1997. Wavelets: A Mathematical Tool for Signal Analysis. SIAM. Philadelphia. PA.

[6] Gao, J., and Y. Jiang. 2008. Trigonometric Hermite wavelet approximation for the integral equations of second kind with weakly singular kernel. Journal of Computational and Applied Mathematics. 215: 242 – 259.

[7] Lepik, Ü., and E. Tamme. 2005. Application of the Haar wavelets for solution of linear integral Equations. Ant. Turk–Dynam. Sys. Appl. Proce. 395–407.

[8] Okayama, T., T. Matsuo, and M. Sugihara. 2010. Sinc-collocation methods for weakly singular Fredholm integral equations of the second kind. Journal of Computational and Applied Mathematics. 234: 1211-1227.

[9] Venkatesh, S. G., S. K. Ayyaswamy, S. R. Balachandar, and K. Kannan. 2013. Wavelet solution for class of nonlinear integro-differential equations. Indian J. Sci.Technol. 6: 4670–4677.